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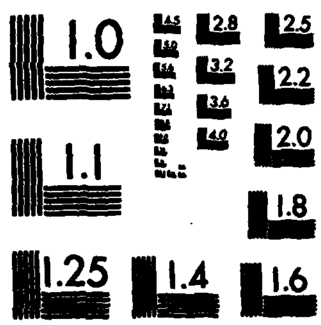


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An effective scheme for subarraying an antenna array is presented. The subarray technique uses a partitioned weighted Butler matrix transform where the partition is an adapting function of the geometry of the noise environment. After subarraying, the subarrayed outputs are adaptively weighted in such a way so as to maximize the output signal-to-noise ratio. It is shown that this technique effectively cancels sidelobe jamming without significantly degrading the signal-to-noise ratio that is possible if the entire array is used for adaptation. In addition it is shown that the

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20. ABSTRACT (Continued)

effective reduction in dimensionality decreases the convergence time of the adapting weights and that the convergence time is independent of the array size.

## CONTENTS

# ADAPTIVE ANTENNA SUBARRAYING USING A WEIGHTED BUTLER MATRIX

## INTRODUCTION

A fully adaptive array is one in which every element of the array is individually controlled adaptively. In theory, this provides the necessary degrees of freedom to lower all of the deterministic sidelobes to any arbitrary level. A partially adaptive array is one in which elements are controlled in groups (the subarray approach) or in which only certain elements called auxiliary elements are made controllable.

The partially adaptive array has not been extensively studied as evidenced by the lack of references available in the literature [1-5]. Chapman [3] initiated one of the earliest studies (circa 1974) into exploring how to effectively combine the $N$ elements of an entire planar antenna array into a collection of $L$ subarrays. He developed the row-column precision antenna array (RCPAA) in which each element signal is split into two paths: a row path and a column path. All the elements of a given row or column are added together, and all the row outputs and column outputs are then adaptively combined. Morgan [2] studied the technique of just using auxiliary elements to perform the adaptation. He showed that the correlation coefficient between the adaptive element spatial vectors plays a key role in characterizing the performance of a partially adaptive array. A significant difference in array performance can occur if the array elements selected for adaptive control are not properly chosen.

Obviously, the fully adaptive configuration is preferred since it offers the most control over the response of the array. However, the typical array may have many elements. This poses several immediate problems. Implementing an adaptive array with many degrees of freedom can have considerable impact on the total cost of the array. This alone, however, is not sufficient grounds for rejecting the idea. Processor implementation does, however, pose a more serious problem for a system of this size. In order to appreciate this, it is necessary to elaborate somewhat upon the ways in which such a processor can be implemented.

For an adaptive processor with $N$ degrees of freedom, each of the $N$ input channels is multiplied by a complex weight and summed to form the output. The adaptive weights are determined according to a control law [6]

$$MW = \mu S, \tag{1}$$

where $M$ is an $N \times N$ matrix of the cross-covariances of the signals in the $N$ channels, $W$ is an $N$-element vector of the weights, $\mu$ is an arbitrary nonzero constant, and $S$ is a vector representation of the desired signal in each channel. Solving for the weights is equivalent to solving a set of $N$ simultaneous linear equations. The most straightforward way in which to implement a processor to do this is to use a form of the Howells control loop in analog hardware [7], or the Widrow LMS algorithm [8], or one of its derivatives in a digital machine. Both are realizations of the "steepest descent" method, an iterative technique employed in the minimization of a function. This is a reliable method of solution in this application since it can be shown that the surface described by the output residue as a function of $W$ is a concave quadratic hypersurface and, as such, contains only one minimum. The problem with this approach is that the convergence rate depends upon the particular physical configuration of the

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interference sources and the array. Mathematically, the transient response of the processor is a function of the eigenvalues of the covariance matrix $M$. The spread in the amplitude of the eigenvalues can be considerable and, since the particular configuration of noise sources to be encountered is never known a priori, the resultant variance in settling time can be large, particularly if the order of $M$ is large, i.e., a large number of degrees of freedom are used.

Reed et al. [9] have proposed to solve directly for the weights by inverting an estimate of the true covariance matrix, thereby eliminating the uncertainty of the convergence rate of the iterative method. This method, too, is subject to limits related to the order of the matrix, $M$. The number of arithmetic steps required to obtain the solution by this method is proportional to the cube of the order, $N$. This, coupled with the requirement for high numerical accuracy, puts an upper bound on $N$ which is related to the computation time available and on the cost that can be tolerated.

In addition, if only auxiliary elements are chosen as inputs to the sidelobe canceller, jammer nulling is effected by the bandwidth-aperture product; i.e., if the number of antenna elements is held constant, as the bandwidth of the jammer or the size of the antenna aperture in terms of wavelengths or the spacing of the auxiliary elements is made larger, the nulling of the jammer is degraded. This degradation results because as auxiliary antenna elements are spaced farther apart, the input noise signal with a spread spectrum becomes more uncorrelated between elements. However, it is the correlation between the signals on the respective antenna elements that allows the adaptive array to sense the direction of arrival of an undesirable signal and then place a null in the antenna pattern in the direction of that undesirable signal.

All of these considerations point toward the desirability of reducing the dimensionality of the processor while maintaining as much control as possible over the size of a given aperture. Reduction of the dimensionality of the processor is achieved through subarraying as illustrated in Fig. 1. Here, the dimensionality is reduced from $N$ to $L$, where $L < N$, by using a transformation matrix $H$ which is a $L \times N$ matrix. It is not a trivial problem to specify an effective $H$ matrix. If the $H$ matrix is specified improperly, the overall signal-to-noise ratio can degrade significantly as a function of the direction of arrival angles of the jamming.

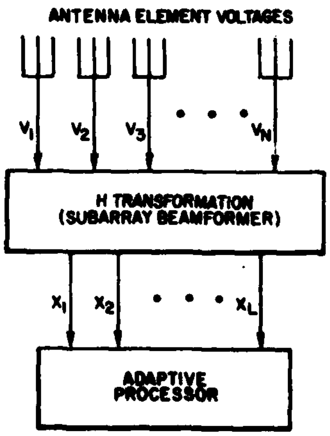


Fig. 1 — Adaptive subarray configuration

This report describes an effective procedure for specifying the $H$ matrix. However, the $H$ matrix will not be a constant as in the RCPAA [3] approach but it will be a function of the noise environment. In a sense, the $H$ matrix will adapt. We will see in the next section that an effective choice for $H$ is a partition of a weighted Butler matrix.

The next section describes the algorithm of specifying $H$ in detail. Then results are presented to support utility of the subarray algorithm. Specifically it is shown that the maximum signal-to-noise ratio possible using *no* subarraying suffers a small degradation if an adaptive weighted Butler matrix subarraying scheme is used. Next, the advantages of using this subarraying scheme are presented.

It should be noted that in this treatment of subarraying for simplicity's sake we deal only with linear antenna arrays. Later reports will consider planar arrays.

## THE WEIGHTED BUTLER MATRIX TRANSFORMATION

Let the transformation, $H$, seen in Fig. 1 be a Butler matrix. A Butler matrix, $B$, has the form

$$B = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \Gamma_N & \Gamma_N^2 & \cdots & \Gamma_N^{N-1} \\ 1 & \Gamma_N^2 & \Gamma_N^{(2)(2)} & \cdots & \Gamma_N^{2(N-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \Gamma_N^{N-1} & \Gamma_N^{2(N-1)} & \cdots & \Gamma_N^{(N-1)(N-1)} \end{bmatrix}, \tag{2}$$

where

$$\Gamma_N = e^{-j\frac{2\pi}{N}} \tag{3}$$

and $j = \sqrt{-1}$. Note that in this case $L = N$. If $\bar{v}$ signifies the input vector and $\bar{u}$ the output vector then $\bar{u} = B\bar{v}$ where $\bar{u} = (u_1, \ldots, u_N)^T$ and $\bar{v} = (v_1, \ldots, v_N)^T$. (Note that we use $\bar{u}$ as the output of the transformation instead of $\bar{x}_N = (x_1, \ldots, x_L)^T$ as depicted in Fig. 1. We do this because we will use a subset of the $u_k$, $k = 1, 2, \ldots, N$ outputs as inputs to the adaptive processor. These inputs will be denoted by $\bar{x}$.) An output, $u_k$, of the transformation has the form

$$u_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} v_n \Gamma_N^{kn}. \tag{4}$$

The form of Eq. (4) is recognized as that of a discrete Fourier transform. It transforms $v_n$ which is a function of the position on the array (array space) indicated by $n$, into angular beam space. Thus, if there is a jammer located $\theta$ degrees off-boresight such that

$$v = (1, e^{j\phi}, e^{2j\phi}, \ldots, e^{(N-1)j\phi})^T \tag{5}$$

where $\phi = (2\pi d/\lambda)\sin\theta$, $d$ is the element separation, and $\lambda$ is the wavelength, then the transformation indicated by Eq. (4) places most of the jammer power into the transformation output, $u_k$, where $k$ is the integer value that makes $|(2\pi k/N) - \phi|$ a minimum.

To illustrate this further, consider the output response of $u_k$ *vs* $\phi$. We can show using Eqs. (4) and (5) that

$$u_k = \frac{\sin^2 \frac{N(\alpha - \phi)}{2}}{\sin^2 \frac{\alpha - \phi}{2}} ; \alpha = \frac{2\pi k}{N}. \tag{6}$$

The output $u_k$ as a function of $\phi$ is shown in Fig. 2. Observe that $u_k$ is relatively large if $\phi$ is close to $(2\pi k/N)$ but decreases rapidly as $|(2\pi k/N) - \phi|$ increases. Hence, the transformation by a Butler matrix tends to orthogonalize the outputs of the matrix transformation, i.e.,

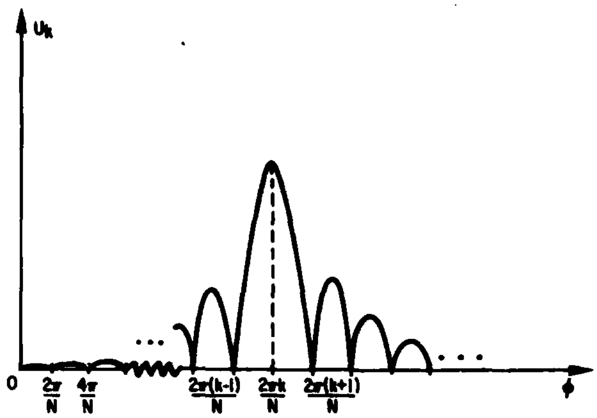


Fig. 2 — $u_k$ *vs* $\phi$

$$\frac{E\{u_i u_k\}}{E\{u_i^2\}} \approx 0 \tag{7}$$

for $i \neq k$. This approximation tends to become better as $|i - k|$ increases.

As has been shown by past researchers [9-15], the orthogonalization (for example Gram-Schmidt orthogonalization developed by Lewis and Kretschmer) of the array inputs by preprocessing enhances the convergence rate of obtaining the optimal adaptive weights. The convergence rate using preprocessing is essentially independent of the eigenvalues of the input covariance matrix and hence, does not suffer the limitation of the control loop implementations. Hence, the settling times of the weights are almost as fast as those obtained by using direct matrix inversion [9].

We can extend the concept of using a Butler matrix transformation to that of using a weighted Butler matrix where

$$H = \frac{1}{\sqrt{N}} \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_N \\ a_1 & a_2\Gamma_N & a_3\Gamma_N^2 & \cdots & a_N\Gamma_N^{N-1} \\ a_1 & a_2\Gamma_N^2 & a_3\Gamma_N^4 & \cdots & a_N\Gamma_N^{2(N-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & a_2\Gamma_N^{N-1} & a_3\Gamma_N^{2(N-1)} & & a_N\Gamma_N^{(N-1)(N-1)} \end{bmatrix}. \tag{8}$$

Note that we can write $H$ as

$$H = BA \tag{9}$$

where $A$ is a diagonal matrix with diagonal elements $a_k$, $k = 1, \ldots, N$. The reason for using the weights, $a_k$, is to control the near-in sidelobe levels of the response of $u_k$ *vs* $\phi$. However, by lowering the near-in sidelobe levels we also enlarge the beamwidth of the response of $u_k$, $k = 1, \ldots, N$ *vs* $\phi$. If this occurs, it may then be desirable to beamform as an output every other $u_k$, or every third $u_k$, etc. Hence, by controlling these sidelobes we can make adjacent outputs more uncorrelated and therefore enhance the convergence rate. However, by doing this we may lose some desired signal power ratio because now $H$ may not be unitary transformation.

For the subarraying technique to be presented, we only use $L$ of the Butler transformed outputs where $L < N$. We choose the $L$ outputs as follows (see Fig. 3): Let $\{x_k\}$, $k = 1, \ldots, L$ be the subset of $\{u_k\}$ $k = 1, \ldots, N$ that is chosen as inputs to the adaptive processor such that $x_1$ is always chosen as one of the $L$ outputs. It can be shown directly that

$$x_1 = \sum_{k=1}^{N} a_k v_k. \quad (10)$$

Thus we use $x_1$ as the main antenna channel where the weights $a_k$, $k = 1, \ldots, N$ can also be specified to control the steering, sidelobe level, beamwidth, ect. of the antenna. Therefore, we see that the adaptive antenna is configured as a sidelobe canceller. After multiplication by the weighted Butler matrix the, $u_k$, $k = 2, \ldots, N$ are ordered from largest to smallest based on their individual powers. Next the $L - 1$, $u_k$, $k = 2, \ldots, L$ voltages with the largest powers are selected as the inputs, $x_k$, $k = 2, \ldots, L$ to the sidelobe canceller.

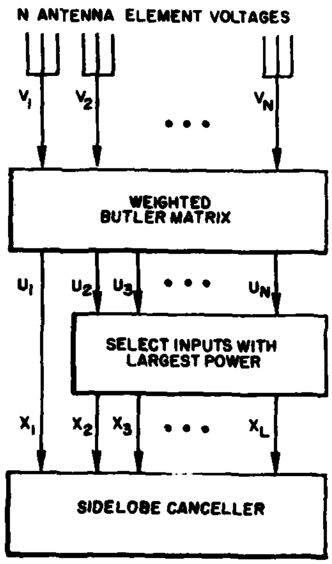


Fig. 3 — Functional block diagram of subarraying algorithm

The largest powers of the outputs were chosen based on the analysis rendered in Appendixes A and B and the next section. There we demonstrate empirically (and analytically in a specific case) that the sidelobe canceller suffers minimum loss in signal-to-noise ratio if the largest powers of the outputs of the weighted Butler matrix transformation are selected.

It is evident from the procedure defined that the number, $L$, of subarray outputs can be a variable. For example, if there are only $N_J$ spatially distinct narrowband jammers then $L = N_J$. Also, if we set a power threshold such that only those outputs, $u_k$, $k = 2, \ldots, N$ that exceed this threshold are chosen, then $L$ will be a function of the threshold, the noise environment, and the sidelobe levels that result from weighting the Butler matrix.

## RESULTS

One of the costs of subarraying is the maximum signal-to-noise ratio that is possible, $(S/N)_{sub}$, degrades from the maximum signal-to-noise ratio, $(S/N)$, that is possible with no subarraying. We define this loss to be

$$(L_{sub})_{dB} = \left| 10 \log \frac{(S/N)_{sub}}{(S/N)} \right|, \tag{11}$$

where $|\cdot|$ denotes absolute value. We demonstrate that using the subarraying scheme described in the previous section for the cases considered results in a small loss.

Let us first consider the following optimization problem: given that there is a narrowband jammer $\theta_J$ degrees off-boresight with jamming power $\hat{\sigma}_J^2$ normalized to the internal noise power, what $2 \times N$ subarray transformation using a partition of the Butler matrix with uniform weights results in the minimum subarray $(S/N)$ loss, $L_{sub}$?

In more mathematical terms, if

$$H = B_p \tag{12}$$

where

$$B_p = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j\psi_k} & e^{-2j\psi_k} & \cdots & e^{-(N-1)j\psi_k} \end{bmatrix}, \tag{13}$$

and

$$\psi_k = \frac{2\pi}{N} \cdot k, \qquad k = 1, 2, \ldots, N-1, \tag{14}$$

then what value of the integer $k$ results in the minimum loss as defined by Eq. (11)? Note $B_p$ is a partition of the Butler matrix, $B$.

For this problem, expressions for $(S/N)_{sub}$ and $(S/N)$ are derived in Appendix A (also in this appendix an equivalent transform to the one seen in Fig. 1 is derived and it is shown in general that for any subarray transformation, $(S/N)_{sub}$ can be found directly from the formulated expression of $(S/N)$). These expressions were used in Appendix B to show that $k$ should be chosen to minimize $|\phi - \psi_k|$, where $\phi = (2\pi d/\lambda)\sin\theta$. Hence, $\psi_k$ should be selected as close to the value of $\phi$ as possible to obtain the minimum $S/N$ loss.

It is also shown in Appendix B that if a $2 \times N$ weighted Butler matrix is used such that $|\phi - \psi_k|$ is a minimum, and $N\hat{\sigma}_J^2 >> 1$ then the maximum $S/N$ loss in dB is approximately

$$(L_{sub}^{MAX})_{dB} = 10.716 ||\tilde{g}(\phi)||^2 \text{ dB}, \tag{15}$$

where

$$\tilde{g}(\phi) = \frac{1}{N} \sum_{k=0}^{N-1} a_{k+1} e^{jk\phi}, \tag{16}$$

and $||\cdot||$ denotes complex magnitude. Note that the quiescent (no jamming) antenna pattern can be defined in terms of $a_k$, $k = 1, \ldots, N$ and is equal to $||\tilde{g}(\alpha)||^2$ if we normalize all antenna gains to the mainbeam gain of a uniformly weighted array. $\alpha = (2\pi d/\lambda)\sin\beta$ and $\beta$ is an arbitrary angle off-boresight. We see from Eq. (15) that the $(S/N)$ loss in dB will be small if the jammer is located in the sidelobes and is proportional to the normalized antenna gain in the direction of the jammer. For example, if the quiescent normalized antenna gain in the direction of the jammer is $-20$ dB, then the $(S/N)$ loss calculated using Eq. (15) is approximately no greater than 0.1 dB.

To further illustrate that choosing the $k$ that minimizes $|\phi - \psi_k|$ results in the minimum $S/N$ loss we have plotted this loss *vs* $k$ and $\psi_k$ $k = 1, 2, \ldots, N-1$ for the following cases:

1. $N = 8$, $L = 2$, $\theta_J = 10°$ (Fig. 4),
2. $N = 16$, $L = 2$, $\theta_J = 10°$ (Fig. 5),
3. $N = 16$, $L = 2$, $\theta_J = 45°$ (Fig. 6).

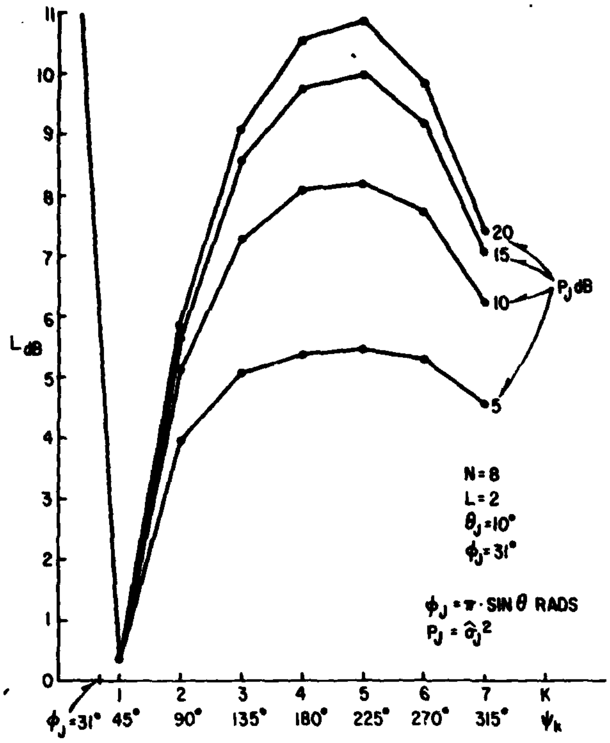


Fig. 4 — Subarray loss *vs* $k$; $N = 8$, $\theta_J = 10°$

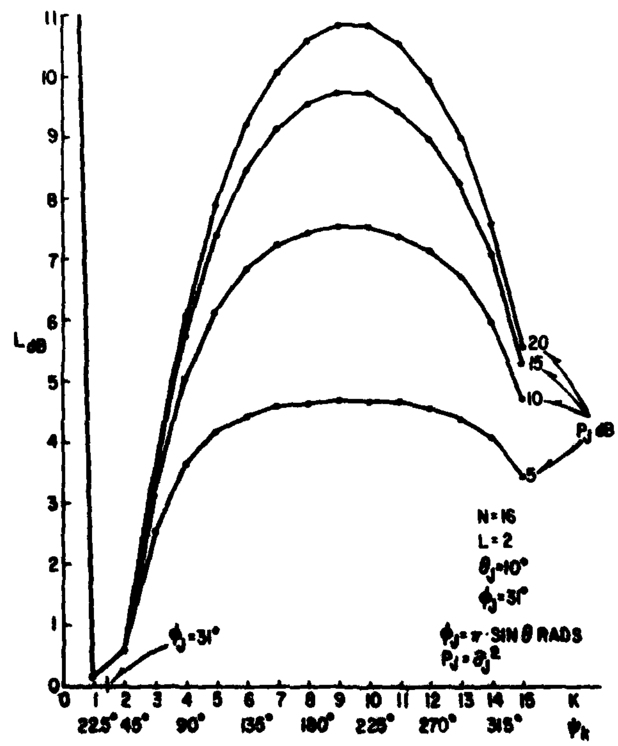


Fig. 5 — Subarray loss *vs* $k$; $N = 16$, $\theta_J = 10°$

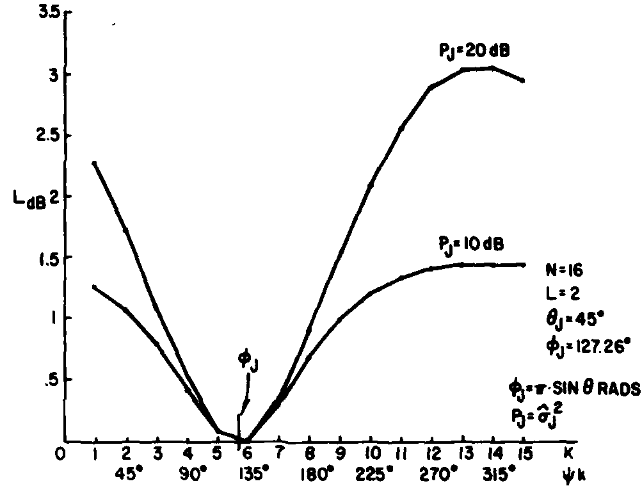


Fig. 6 — Subarray loss *vs* $k$; $N = 16$, $\theta_J = 45°$

In these cases, a uniformly weighted Butler matrix is used. We see from Figs. 4, 5, and 6 that indeed the minimum $(S/N)$ loss occurs when $|\phi - \psi_k|$ is a minimum. For cases 1 and 2, $\phi = 31°$ and for case 3, $\phi = 127°$. Note from comparing the curves of Fig. 5 with Fig. 6, that the minimum loss is greater as the jammer's direction of arrival (DOA) moves closer to boresight. This observation is predicted by Eq. (15).

Now consider the case where there are two jammers and the number of antenna elements is 36. Both jammers have $\hat{\sigma}_J^2 = 20$ dB power and the first jammer is $\theta_1 = 25°$ off-boresight. Let us vary the DOA, $\theta_2$, of the second jammer from 0° to 90° off-boresight and subarray using a $3 \times N$ uniformly weighted partitioned Butler matrix, such that

$$B_p = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j\psi_{k_1}} & e^{-2j\psi_{k_1}} & \cdots & e^{-(N-1)j\psi_{k_1}} \\ 1 & e^{-j\psi_{k_2}} & e^{-2j\psi_{k_2}} & \cdots & e^{-(N-1)j\psi_{k_2}} \end{bmatrix}. \tag{17}$$

In the above equation $\psi_{k_1} = (2\pi/N)k_1$, $\psi_{k_2} = (2\pi/N)k_2$ and $k_1, k_2$ are chosen to minimize $|\phi_1 - \psi_{k_1}|$ and $|\phi_2 - \psi_{k_2}|$ ($\phi_1 = 31°$, $\phi_2 = \pi \sin\theta_2$) such that $k_1 \neq k_2$ (note if $k_1 = k_2$ then the $H$ transformation is of rank 2 which renders the calculation of the $(S/N)$ loss impossible because of the necessity of inverting a singular matrix; see Appendix A, Eq. (A5)). If it results that $k_1 = k_2$, then we choose $k_2 = k_1 + 1$ or $k_2 = k_1 - 1$ such that $k_2 \neq 0$ MOD $(N)$.

The resultant $(S/N)$ loss is graphed in Fig. 7. We have not attempted to plot $L_{dB}$ for every $\theta_2$ but have plotted $L_{dB}$ in increments of $\Delta\theta_2 = 1°$. First note that in the sidelobes for this case that the average loss is small ($\approx 0.008$ dB). Secondly, the loss becomes larger as the second jammer moves through the boresight of the antenna. This is not unexpected because the null in the direction of the jammer also nulls the desired signal as the null is placed closer to the main beam. Hence a loss of $(S/N)$ performance occurs.

The undulating nature of the subarray loss is due to the undulating nature of $|\phi_2 - \psi_{k_2}|$. A plot of $|\phi_2 - \psi_{k_2}|$ appears in Fig. 8. As $|\phi_2 - \psi_{k_2}|$ increases, the subarray loss becomes larger because the phase, $\phi_2$, is not centered on one of the discrete angles, $(2\pi/N) \cdot k$, $k = 1, 2, \ldots, N-1$. Also note

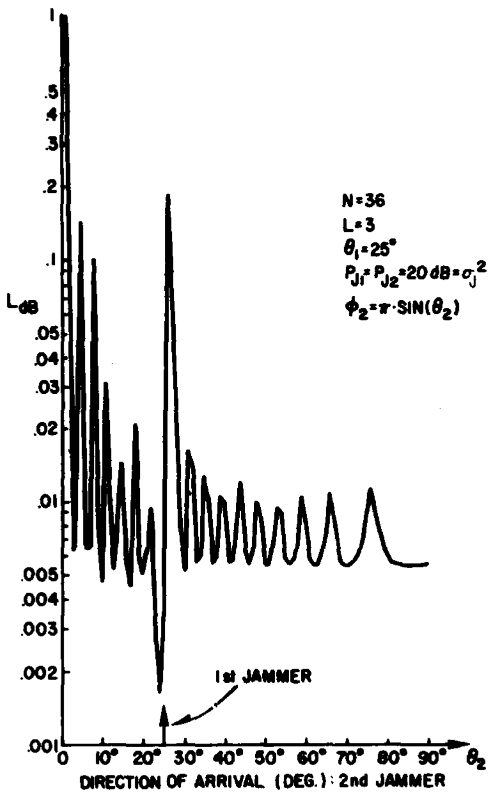


Fig. 7 — Subarray loss vs $\theta_2$; $\theta_1 = 25°$, $N = 36$, $L = 3$.

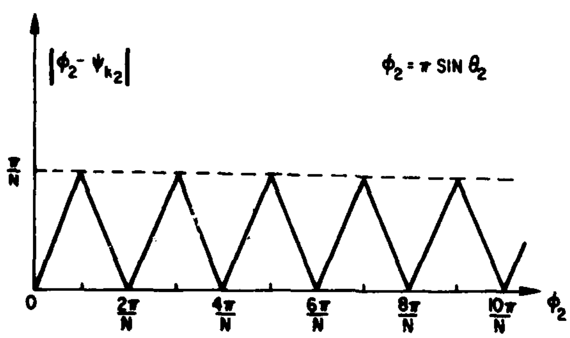


Fig. 8 — $|\phi_2 - \psi_{k_2}|$ vs $\phi_2$

in Fig. 7 that as the second jammer angle passes thru the first jammer angle ($\theta_2 \approx 25°$), there is at first a very small $(S/N)$ loss ($\approx 0.0016$ dB) but then a noticeably large $(S/N)$ loss ($\approx 0.19$ dB). We will explain this phenomenon in a future report.

## ADVANTAGES AND FUTURE STUDIES

Some of the major advantages of subarraying as described in the preceding sections are that:

1. it reduces the dimensionality of the sidelobe canceller processor which reduces the amount of necessary hardware,

2. it effectively cancels sidelobe jamming without significant degradation of the output signal-to-noise ratio,

3. reduction of the dimensionality of the sidelobe canceller decreases the convergence time of the sidelobe canceller (for the matrix inversion algorithm [1] convergence rate is proportional to the dimensionality),

4. convergence time of the sidelobe canceller is independent of the size of the array; it is proportional to the number of jammers,

5. orthogonalization of the channels allows the theoretical convergence time to be achieved since the loops are now almost independent and do not interact,

6. the effect of the bandwidth-aperture product is reduced since all the elements of the antenna are used to form the reduced number of inputs to the sidelobe canceller.

The third advantage mentioned above results because if the direct matrix inversion algorithm [9] is used to compute the optimal weights, then the average number of samples of each subarray output required, such that the average output signal-to-noise ratio is within 3 dB of the optimum subarrayed signal-to-noise ratio, is equal to $2L$ where $L$ is the number of subarrayed outputs. If we do not use subarraying but use the entire array then the number of samples of each array output needed is $2N$. Obviously if $L << N$, then the number of samples needed when subarraying is much less when not subarraying. Hence, the convergence rate will be faster. (Note that it is as assumed that the samples are from time independent zero-mean Gaussian processes.)

The fourth advantage is true under the assumption that we are thresholding the powers of the individual output channels of the weighted Butler transformation.

Future studies will be directed into investigating the effect of random errors in the antenna elements. We have assumed in our analysis that all of the antenna elements are identical. In practical cases this is not true. There will be small random phasor errors which multiply the outputs of the individual antenna elements. These random errors will degrade $S/N$ performance and upper bound the maximum $S/N$.

A second study will be undertaken to investigate the effects of the jammer bandwidth-antenna aperture product on the subarry's performance. Since this product inherently limits the performance of a totally adaptive array, it may or may not further limit the performance of a subarrayed adaptive antenna where we subarray using a weighted Butler matrix.

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## Appendix A

## EQUIVALENT TRANSFORMATIONS FOR SUBARRAYS

In this appendix, we prove that the subarraying transformation as seen in Fig. A1 is equivalent to the transformation seen in Fig. A2 in that the resultant maximum signal-to-noise ratios are the same. The $H$ transformation seen in Fig. A1 is an $L \times N$ matrix and has rank $L$.

The transformation viewed in Fig. A1 works as follows: the internal (thermal noise) and external inputs add at the front end (the antenna elements) after which an $L \times N$ matrix transformation reduces the $N$ inputs to $L$ outputs ($L < N$). Finally, these $L$ outputs are optimally weighted such that the maximum signal-to-noise ratio is achieved.

The transformation as seen in Fig. A2 is not actually realizable because here we are separating the internal noises from the external noises and operating on them separately. This transformation works as follows: the $N$ external inputs are multiplied by an $N \times N$ matrix $P$ where $P = H^t(HH^t)^{-1} H$ and $t$ denotes conjugate transpose. The outputs of this transformation are then added to the internal inputs (thermal noise of the same rms power as the thermal noise seen in Fig. A1). Finally, the resultant $N$ outputs are optimally weighted.

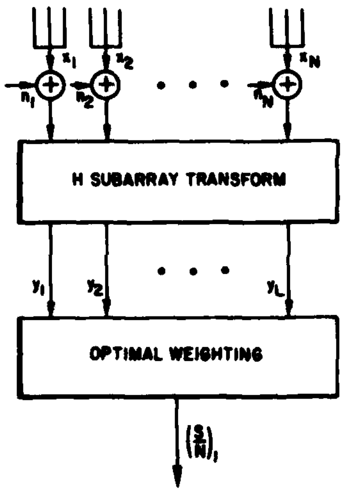


Fig. A1 — Subarray transformation

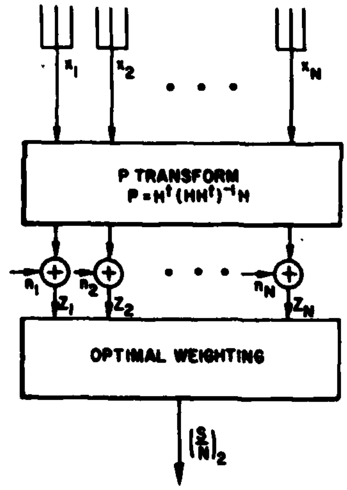


Fig. A2 — Equivalent transformation

Let $\overline{X}$ represent the noise vector of $N$ external antenna inputs, $\overline{\eta}$ represent the vector of $N$ internal noise sources, and $\overline{V}$ represent the total noise vector. Obviously

$$\overline{V} = \overline{X} + \overline{\eta}, \tag{A1}$$

and

$$E\{\overline{V}\overline{V}^t\} = E\{\overline{X}\overline{X}^t\} + E\{\overline{\eta}\overline{\eta}^t\}, \tag{A2}$$

assuming the internal noise processes are zero mean and independent of the external noises. Furthermore, let the internal noise sources be statistically independent and identically distributed with noise power equal to $\sigma_0^2$. Thus

$$E\{\bar{\eta}\bar{\eta}'\} = \sigma_0^2 I \tag{A3}$$

where $I$ is the $N \times N$ identity matrix. If we set $M = E\{\bar{V}\bar{V}'\}$ and $M_J = E\{\bar{X}\bar{X}'\}$, then

$$M = \sigma_0^2 I + M_J. \tag{A4}$$

If $S$ represents the input signal vector, then it can be shown [A1] that the maximum signal-to-noise ratio, $(S/N)_1$ that is possible by properly weighting the outputs, $\bar{Y}$, of the subarray transformation as seen in Fig. A1 is

$$\left(\frac{S}{N}\right)_1 = S'H'(HMH')^{-1}HS. \tag{A5}$$

We can also show that the maximum signal-to-noise ratio, $(S/N)_2$, that is possible by properly weighting the output, $\bar{Z}$, as seen in Fig. A2 is

$$\left(\frac{S}{N}\right)_2 = S_p'M_p^{-1}S_p, \tag{A6}$$

where

$$S_p = PS, \tag{A7}$$

$$M_p = \sigma_0^2 I + PM_JP', \tag{A8}$$

and

$$P = H'(HH')^{-1}H. \tag{A9}$$

We will show that $(S/N)_1 = (S/N)_2$.

To this end, let us normalize the covariance matrices to the internal noise power such that

$$\hat{M} = I + \hat{M}_J, \tag{A10}$$

and

$$\hat{M}_J = \frac{1}{\sigma_0^2} M_J. \tag{A11}$$

It is easy to show that

$$\left(\frac{S}{N}\right)_1 = \frac{1}{\sigma_0^2} S'H'(H\hat{M}H')^{-1}HS, \tag{A12}$$

and

$$\left(\frac{S}{N}\right)_2 = \frac{1}{\sigma_0^2} S_p'\hat{M}_p^{-1}S_p, \tag{A13}$$

where

$$\hat{M}_p = I + P\hat{M}_JP'. \tag{A14}$$

We write $(S/N)_2$ in a more expanded form

$$\left(\frac{S}{N}\right)_2 = \frac{1}{\sigma_0^2} S'P'\hat{M}_p^{-1}PS. \tag{A15}$$

We prove that $(S/N)_1 = (S/N)_2$ by showing that

$$H'(H\hat{M}H')^{-1}H = P'\hat{M}_p^{-1}P. \tag{A16}$$

Before we do this, let us state that it can be shown that $P$ is an idempotent matrix, i.e., $P^2 = P$ and therefore singular for $N \geqslant 2$. In addition $P$ is hermitian, $P = P'$.

We prove the identity of Eq. (A16) by considering the left-hand side of the equation. Now using (A10)

$$(H\hat{M}H')^{-1} = (HH' + H\hat{M}_J H')^{-1}. \tag{A17}$$

Invoking the matrix inversion lemma [A2] we can show

$$(HH' + H\hat{M}_J H')^{-1} = (HH')^{-1} - (HH')^{-1}H\hat{M}_J(I + H'(HH')^{-1}H\hat{M}_J)^{-1}H'(HH')^{-1} \tag{A18}$$

Hence, using Eq. (A9) it follows that

$$H'(HH' + H\hat{M}_J H')^{-1}H = P - P\hat{M}_J(I + P\hat{M}_J)^{-1}P. \tag{A19}$$

Now consider the right-hand side of Eq. (A16). Equation (A14) implies that

$$\hat{M}_p^{-1} = (I + P\hat{M}_J P')^{-1}. \tag{A20}$$

Again using the matrix inversion lemma, we can show

$$(I + P\hat{M}_J P')^{-1} = I - P\hat{M}_J(I + P'P\hat{M}_J)^{-1}P'. \tag{A21}$$

Since $P'P = P$, $P^2 = P$, and $P = P'$ then

$$P(I + P\hat{M}_J P')^{-1}P' = P - P\hat{M}_J(I + P\hat{M}_J)^{-1}P'. \tag{A22}$$

Comparing Eq. (A19) with Eq. (A22), we see that Eq. (A16) follows, and thus, $(S/N)_1 = (S/N)_2$. Hence, the configurations of Figs. A1 and A2 are equivalent in that the resultant maximum signal-to-noise ratios are identical.

This equivalency can be effectively utilized as follows. Suppose there are $K$ external noise sources with respective direction-of-arrival vectors (DOAV), $A_k$, and powers $\sigma_k^2$, $k = 1, \ldots, K$. If there is no subarraying as seen in Fig. A1 we can derive an expression for the optimal signal-to-noise ratio, $(S/N)$, which is a function of $A_k$, $\sigma_k^2$, $k = 1, \ldots, K$ and the steering vector $S$ such that

$$\left(\frac{S}{N}\right) = F(A_1, \ldots, A_K, \sigma_1^2, \ldots, \sigma_K^2, \sigma_0^2, S), \tag{A23}$$

where $F$ is the functional relationship between the inputs and $S/N$. In order to find the maximum signal-to-noise ratio $(S/N)_1$ of an array that has a subarray transformation we use the equivalency of the configuration in Fig. A2 to show that

$$\left(\frac{S}{N}\right)_1 = F(PA_1, \ldots, PA_K, \sigma_1^2, \ldots, \sigma_K^2, \sigma_0^2, PS). \tag{A24}$$

Equation (A24) follows because the $(S/N)$ as expressed by Eq. (A23) holds for any $K$ arbitrary external input noise vectors, steering vector, and noise powers. Hence, after the transformation by $P$, the external input noise vectors are $PA_k$, $k = 1, \ldots, K$, the external input noise powers are unchanged, and the steering vector is $PS$. For example, if there is only one narrowband noise source with DOAV, $A_1$, and power $\sigma_1^2$, we can show that if subarraying is *not* used, then

$$\left(\frac{S}{N}\right)_{opt} = \frac{1}{\sigma_0^2}\left(||S||^2 - \frac{\sigma_1^2}{\sigma_0^2 + \sigma_1^2 A_1' A_1}\, ||S' A_1||^2\right). \tag{A25}$$

where $||\cdot||$ denotes complex magnitude. Now if we place a subarray transformation, $H$, between the antenna elements and the optimal weights, Eq. (A24) implies that in order to find the optimal subarrayed signal-to-noise ratio, we replace $A_1$ with $PA_1$ and $S$ with $PS$ in Eq. (A24). This results in

$$\left(\frac{S}{N}\right)_1 = \frac{1}{\sigma_0^2}\left(||PS||^2 - \frac{\sigma_1^2}{\sigma_0^2 + \sigma_1^2 A_1' PA_1}\, ||S' PA_1||^2\right). \tag{A26}$$

To derive $(S/N)_1$ in Eq. (A26), we used the fact that $PP' = P$.

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## Appendix B

## OPTIMAL 2 × N BUTLER SUBARRAY TRANSFORMATION

In this appendix, we find the optimal $2 \times N$ Butler subarray transformation. The transformation is optimal in the sense that the signal-to-noise ratio loss that is incurred by subarraying using a $2 \times N$ partition of a Butler matrix is minimized. We also derive suboptimal $2 \times N$ weighted Butler matrix transformations where it is shown that subarraying loss, although not optimal, is small and within easily computed bounds.

Let us define a subarray transformation, $H$, as follows:

$$H = B_p \tag{B1}$$

where

$$B_p = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j\psi_n} & e^{-2j\psi_n} & \cdots & e^{-(N-1)j\psi_n} \end{bmatrix}, \tag{B2}$$

and

$$\psi_n = \frac{2\pi}{N} \cdot n \qquad n = 1, 2, \ldots, N-1. \tag{B3}$$

*Note that $B_p$ is a partition* of a Butler matrix. We are given that the $N$ length signal vector, $S$, is defined as

$$S = (1 \;\; 1 \;\; \ldots \; 1)^T \sigma_S^2 \tag{B4}$$

for an uniformly weighted array, which indicates that the mainbeam is pointing in the direction of boresight. In Eq. (B4), $\sigma_S^2$ is the received power of the desired signal at each antenna element. Note that multiplying the $N$ array antenna inputs by $H$ results in two channels. One channel will be the main channel in that it is matched to the steering vector. In the main channel, energy received at an angle $\beta$ off boresight will have the following gain (or antenna pattern):

$$g(\alpha) = \sum_{k=0}^{N-1} e^{jk\alpha}, \tag{B5a}$$

where $\alpha = (2\pi d/\lambda_0)\sin\beta$, $d$ is the linear array spacing, and $\lambda_0$ is the wavelength. The auxiliary channel (the second channel) will have the antenna pattern

$$g_a(\alpha) = \sum_{k=0}^{N-1} e^{jk(\alpha-\psi_n)}, \tag{B5b}$$

$$= g(\alpha - \psi_n).$$

If there is one external jammer with power $\sigma_J^2$ and direction-of-arrival vector

$$A = (1, e^{j\phi}, e^{2j\phi}, \ldots, e^{(N-1)j\phi})^T, \tag{B6}$$

where $\phi = (2\pi d/\lambda_0)\sin\theta$ and $\theta$ is the angle off-boresight of the jammer, we would like to know for what value of $\psi_n$ is the output signal-to-noise ratio maximized. We will show that $\psi_n$ should be chosen as close to the value of $\phi$ as possible. Also, we show that for arbitrary $N$, choosing $\psi_n$ in this fashion results in a small signal-to-noise ratio loss from the maximum signal-to-noise ratio that is possible if no subarraying is used.

Let us normalize the jammer noise power to the internal noise power, $\sigma_0^2$:

$$\hat{\sigma}_J^2 = \frac{\sigma_J^2}{\sigma_0^2}. \tag{B7}$$

We use the result, Eq. (A25), of Appendix A to show that the maximum signal-to-noise ratio, $(S/N)$, possible if subarraying is not used is

$$\left(\frac{S}{N}\right) = \frac{||S||^2}{\sigma_0^2}\left(1 - \frac{\hat{\sigma}_J^2}{1 + ||A||^2\hat{\sigma}_J^2}\,\frac{||S'A||^2}{||S||^2}\right). \tag{B8}$$

Substituting $||S||^2 = N\sigma_S^2$, $||A||^2 = N$, and $S'A = g(\phi)\sigma_S$ we find that

$$\left(\frac{S}{N}\right) = \frac{N\sigma_S^2}{\sigma_0^2}\left(1 - \frac{\hat{\sigma}_J^2}{1 + N\hat{\sigma}_J^2}\,\frac{||g(\phi)||^2}{N}\right). \tag{B9}$$

We see that for $N\hat{\sigma}_J^2 >> 1$ and $||g(\phi)||^2 << N^2$ which is true if the jammer is in the sidelobes, then

$$\left(\frac{S}{N}\right) \doteq \frac{N\sigma_S^2}{\sigma_0^2}. \tag{B10}$$

If subarraying is used, then using Eq. (A26), the maximum attainable signal-to-noise ratio, $(S/N)_{\text{sub}}$ is

$$\left(\frac{S}{N}\right)_{\text{sub}} = \frac{||PS||^2}{\sigma_0^2}\left(1 - \frac{\hat{\sigma}_J^2}{1 + ||PA||^2\hat{\sigma}_J^2}\,\frac{||(PS)'(PA)||^2}{||PS||^2}\right) \tag{B11}$$

where

$$P = H'(HH')^{-1}H = B_p'B_p. \tag{B12}$$

It is straightforward to show that

$$PS = S, \tag{B13}$$

$$||PS||^2 = N\sigma_S^2, \tag{B14}$$

and

$$PA = \frac{1}{N}g(\phi)\bar{1} + \frac{1}{N}g(\phi - \psi)\Lambda_\psi\bar{1}, \tag{B15}$$

where $\bar{1}$ is an $N$ length vector such that

$$\bar{1} = (1\ \ 1 \ldots 1)^T \tag{B16}$$

and $\Lambda_\psi$ is an $N \times N$ diagonal matrix such that $\Lambda_\psi = (\lambda_{kk})$ and

$$\lambda_{kk} = e^{j(k-1)\psi}. \tag{B17}$$

Using Eq. (B15) we can show that

$$\|PA\|^2 = \frac{1}{N^2}\,[\|g(\phi)\|^2 N + g^*(\phi)g(\phi-\psi)\bar{1}'\Lambda_\psi\bar{1} \tag{B18}$$

$$+\, g(\phi)g^*(\phi-\psi)\bar{1}'\Lambda_\psi^*\bar{1}$$

$$+\, \|g(\phi-\psi)\|^2\bar{1}'\Lambda_\psi^*\Lambda_\psi\bar{1}]$$

$$= \frac{1}{N}(\|g(\phi)\|^2 + \|g(\phi-\psi)\|^2)$$

$$+\, \frac{2}{N^2}\,\mathrm{Re}\{g^*(\phi)g(\phi-\psi)g(\psi)\},$$

and that

$$\|(PS)'(PA)\|^2 = \Bigg[\|g(\phi)\|^2 + \frac{1}{N^2}\|g(\psi)\|^2\|g(\phi-\psi)\|^2 \tag{B19}$$

$$+\, \frac{2}{N}\,\mathrm{Re}\{g^*(\phi)g(\psi)g(\phi-\psi)\}\Bigg]\cdot\sigma_S^2.$$

If we set $\psi = \psi_n$ where $\psi_n$ is defined by Eq. (B3), then $g(\psi_n) = 0$ and we can write $(S/N)_{\mathrm{sub}}$ using Eqs. (B14), (B18), and (B19) as

$$\left(\frac{S}{N}\right)_{\mathrm{sub}} = \frac{N\sigma_S^2}{\sigma_0^2}\left(1 - \frac{\|g(\phi)\|^2}{N\hat{\sigma}_J^{-2} + \|g(\phi)\|^2 + \|g(\phi-\psi_n)\|^2}\right). \tag{B20}$$

We see that maximizing $(S/N)_{\mathrm{sub}}$ as a function of $\psi_n$ is equivalent to minimizing the second term of Eq. (B20). This term is minimized when $\|g(\phi-\psi_n)\|^2$ is maximized. This occurs due to the nature of $\|g(\alpha)\|^2 = (\sin^2 N\alpha/2)/\sin^2(\alpha/2)$ when $\psi_n$ is chosen to be the nearest value of $\psi_n$, $n = 1, \ldots, N-1$, to $\phi$. Let this value be $\psi_{\max}$.

In order to evaluate the signal-to-noise ratio loss incurred by subarraying we define

$$L_{\mathrm{sub}} = \frac{(S/N)_{\mathrm{sub}}}{(S/N)}. \tag{B21}$$

Using (B10) and (B20) we see that for $N\hat{\sigma}_J^2 >> 1$

$$L_{\mathrm{sub}} = 1 - \frac{\|g(\phi)\|^2}{N\hat{\sigma}_J^{-2} + \|g(\phi)\|^2 + \|g(\phi-\psi_{\max})\|^2}. \tag{B22}$$

The maximum loss will occur when $\phi$ is located equidistant between $\psi_k$ and $\psi_{k+1}$. This occurs because $\|g(\phi-\psi_{\max})\|^2$ is monotonically decreasing in the region $\psi_{k-1} < \phi < \psi_{k+1}$ where $\psi_k = \psi_{\max}$. Hence, since $\phi - \psi_{\max}$ can be at most $\pi/N$ and $L_{\mathrm{sub}}$ decreases as $\|g(\phi-\psi_{\max})\|^2$ decreases, then the maximum loss occurs when

$$\phi - \psi_{\max} = \frac{\pi}{N}. \tag{B23}$$

Therefore,

$$\|g(\phi-\psi_{\max})\|^2 = \frac{\sin^2\frac{\pi}{2}}{\sin^2\frac{\pi}{2N}} \doteq \frac{4N^2}{\pi^2}. \tag{B24}$$

In Eq. (B24), we assumed that $\pi/2N << 1$. Hence,

$$L_{sub}^{MAX} = 1 - \frac{||g(\phi)||^2}{||g(\phi)||^2 + N\hat{\sigma}_J^{-2} + \frac{4N^2}{\pi^2}}. \tag{B25}$$

We see that if the jammer is in the far sidelobes then $4N^2/\pi^2 >> ||g(\phi)||^2$ and

$$L_{sub}^{MAX} = 1 - \frac{||g(\phi)||^2}{N\hat{\sigma}_J^{-2} + \frac{4N^2}{\pi^2}}. \tag{B26}$$

Equation (B26) implies that the signal-to-noise ratio loss will be small. In order to calculate the maximum loss in dB we use the approximation that for small $\epsilon$

$$10 \log_{10}(1 - \epsilon) = -\frac{10}{\ln 10}\epsilon \text{ dB}.$$

Hence,

$$(L_{sub}^{MAX})_{dB} = \frac{10}{\ln 10} \frac{||g(\phi)||^2}{N\hat{\sigma}_J^{-2} + \frac{4N^2}{\pi^2}} \text{ dB} \tag{B27}$$

$$= \frac{10}{\ln 10} \frac{||\tilde{g}(\phi)||^2}{(N\hat{\sigma}_J^2)^{-1} + \frac{4}{\pi^2}} \text{ dB}$$

where we have normalized $||\tilde{g}(\phi)||^2$ to have a maximum value of one. More explicitly

$$||\tilde{g}(\phi)||^2 = \frac{1}{N^2} \frac{\sin^2 \frac{N\phi}{2}}{\sin^2 \frac{\phi}{2}}. \tag{B28}$$

For $N\hat{\sigma}_J^2 << 4/\pi^2$, Eq. (B28) reduces to

$$(L_{sub}^{MAX})_{dB} = 10.716\, ||\tilde{g}(\phi)||^2 \text{ dB}. \tag{B29}$$

The above results can easily be extended to include steering the mainbeam off-boresight and weighting the antenna elements to obtain a given level of sidelobes, beamwidth, etc. However, the adaptive array with no subarraying must also have the same quiescent antenna pattern determined by the quiescent weighting when calculating the subarray signal-to-noise ratio loss.

In addition, if

$$g(\psi) = \sum_{k=0}^{N-1} a_{k+1} e^{jk\psi}, \tag{B30}$$

where $a_k$ are the quiescent weights which determine the antenna steering, sidelobe level, etc., then $g(\psi_n)$ is not necessarily equal to zero and hence Eq. (B20) is not exactly true. However, we can show that the terms that went to zero in Eqs. (B18) and (B19) are small compared to the other terms so that Eq. (B20) is approximately equal to the maximum signal-to-noise ratio.

In any case, if $\psi_n$ is chosen to be closest to $\phi$, then Eq. (B29) follows where

$$\tilde{g}(\phi) = \frac{1}{N} \sum_{k=0}^{N-1} a_{k+1} e^{jk\phi}. \tag{B31}$$

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